Research Announcement

Continuous q-Legendre Polynomials are Spherical Matrix Elements of Irreducible Representations of the Quantum $SU(2)$ Group

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Let $0<\mu<1$. Let $\mathcal{Q}$ be the Hopf $\star$-algebra corresponding to the quantum $SU(2)$ group. This is the algebra with unit I generated by α, β, γ, δ with relations

$$\begin{aligned} \alpha\beta &= \mu\beta\alpha,\ \alpha\gamma=\mu\gamma\alpha,\ \beta\delta=\mu\delta\beta,\ \gamma\delta=\mu\delta\gamma, \\ \beta\gamma &= \gamma\beta,\ \alpha\delta-\delta\alpha=(\mu-\mu^{-1})\beta\gamma,\ \alpha\delta-\mu\beta\gamma=I, \end{aligned}$$

which is made into a $\star$-algebra by putting $\alpha^{*}=\delta$, $\beta^{*}=-\mu\gamma$, and which is made into a Hopf algebra with comultiplication Φ by requiring that $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ is a matrix corepresentation of $\mathcal{Q}$. See Woronowicz [1].

The irreducible unitary corepresentations of $\mathcal{Q}$ have been classified. For each finite dimension $2l+1$ there is, up to equivalence, precisely one such corepresentation t^{l}. Usually this is written as a matrix corepresentation (t^{l}_{nm}) $(n,m=-l,-l+1,\ldots,l)$ with respect to a basis of eigenvectors for the subgroup $U(1)$, for which the group algebra occurs by adding the relations $\beta=\gamma=0$ to $\mathcal{Q}$. These matrix elements can be expressed in terms of little q-Jacobi polynomials. In particular, for integer l, the matrix element t^{l}_{00} with respect to the $U(1)$-invariant basis vector is the little q-Legendre polynomial of degree l of argument $-\mu^{-1}\beta\gamma$, with base $q=\mu^{2}$. See Vaksman and Soibelman [2], Masuda et al. [3], Koornwinder [4]. In the following I will refer to the explicit expressions for the matrix elements t^{l}_{nm} as given in [4, Theorem 5.3].

In the case of the ordinary group $SU(2)$ the subgroup $SO(2)$ rather than the diagonal subgroup $U(1)$ is also a natural choice for the compact subgroup with respect to which we define the notion of spherical matrix element. In fact, all such subgroups (fixed point groups under an involution) are conjugate in the case of $SU(2)$ and all choices yield the same Legendre polynomials as spherical matrix elements of irreducible representations. In the case of quantum $SU(2)$ there is no quantum subgroup corresponding to $SO(2)$ for $\mu=1$. However, bi-invariance with respect to $SO(2)$ can also be described infinitesimally and this generalizes to the quantum case.

In [2] it was pointed out how Jimbo's quantized universal enveloping algebra in the case of root system A_1 is a dual Hopf algebra for the Hopf algebra $\mathcal{A}$ described above. Here I use another notation. Let $\mathcal{U}$ be a Hopf algebra with unit element 1 and with generators A,B,C,D satisfying the relations

$$AD=DA=1,\ AB=\mu BA,\ AC=\mu^{-1}CA,\ BC-CB=\frac{A^2-D^2}{\mu-\mu^{-1}}.$$

This is made into a Hopf algebra by putting for the comultiplication Δ:

$$\Delta(A)=A\otimes A,\ \Delta(D)=D\otimes D,\ \Delta(B)=A\otimes B+B\otimes D,$$
$$\Delta(C)=A\otimes C+C\otimes D.$$

There is a bilinear nondegenerate pairing between $\mathcal{U}$ and $\mathcal{A}$ such that, for $X, Y\in\mathcal{U}$ and $a,b\in\mathcal{A}$

$$(XY)(a) = (X\otimes Y)(\Phi(a)),\ (\Delta(X))(a\otimes b)=X(ab),$$

co-unit and unit of $\mathcal{A}$ are, respectively, unit and co-unit of $\mathcal{U}$, and,

$$A(\alpha)=\mu^{1/2},\ A(\delta)=\mu^{-1/2},\ D(\alpha)=\mu^{-1/2},\ D(\delta)=\mu^{1/2},$$
$$B(\beta)=1,\ C(\gamma)=1,$$

and $X(a)=0$ for other cases that X is generator of $\mathcal{U}$ and a is generator of $\mathcal{A}$.

Define, for $X\in\mathcal{U}$ and $a\in\mathcal{A}$ the elements $X.a$ and $a.X$ of $\mathcal{A}$ by

$$X.a := (\mathrm{id}\otimes X)(\Phi(a)),\ a.X:=(X\otimes \mathrm{id})(\Phi(a)).$$

Define an element a of $\mathcal{A}$ to be *spherical* if

$$(\mu^{1/2}B-\mu^{-1/2}C).a=0=a.(\mu^{1/2}B-\mu^{-1/2}C).$$

Put

$$\rho := 1/2(\alpha^2+\mu^{-1}\beta^2+\mu\gamma^2+\delta^2).$$

Put, for $l=0,1,2,\ldots$ and $n=-l,\ -l+2,\ldots,l$:

$$c_n^l := \left[\frac{(\mu^2;\mu^4)_{(l-n)/2}(\mu^2;\mu^4)_{(l+n)/2}}{(\mu^4;\mu^4)_{(l-n)/2}(\mu^4;\mu^4)_{(l+n)/2}}\right]^{\frac{1}{2}}.$$

The *continuous q-Legendre polynomial* is the continuous q-ultraspherical polynomial of order $\beta=q^{1/2}$:

$$C_n(\cos\theta;q^{1/2}\,|\,q) := \sum_{k=0}^{n}\frac{(q^{1/2};q)_k(q^{1/2};q)_{n-k}}{(q;q)_k(q;q)_{n-k}}e^{i(n-2k)\theta}$$
$$= q^{-n/4}\,{}_4\Phi_3\left[\begin{matrix}q^{-n},q^{n+1},q^{1/4}e^{i\theta},q^{1/4}e^{-i\theta}\\ q,\ -q,\ -q^{1/2}\end{matrix};q,q\right].$$

THEOREM 1. *Let $a \in \mathcal{A}$. Then a is spherical if and only if it is a polynomial in ρ.*

THEOREM 2. *Fix $l = 0, 1/2, 1, \ldots$. Then the space of spherical elements in the span of the t^l_{nm}, $n,m = -l, -l+1, \ldots, l$ has dimension 0 if l is a half integer and dimension 1 if l is an integer. In the latter case the space is spanned by*

$$C_l(\rho;\mu^2 \mid \mu^4) = \sum_{\substack{n,m=-l \\ l-n,\, l-m \text{ even}}}^{l} \mu^{(n-m)/2} c^l_n c^l_m t^l_{nm}.$$

THEOREM 3. *Let p be a polynomial. Let h be the Haar functional on $\mathcal{A}$ (cf. [1]). Then*

$$h(p(\rho)) = \frac{(\mu^4;\mu^4)_\infty (\mu^4;\mu^4)_\infty}{2\pi(\mu^2;\mu^4)_\infty (\mu^6;\mu^4)_\infty} \int_0^\pi p(\cos\theta) \frac{(e^{2i\theta};\mu^4)_\infty (e^{-2i\theta};\mu^4)_\infty}{(\mu^2 e^{2i\theta};\mu^4)_\infty (\mu^2 e^{-2i\theta};\mu^4)_\infty} d\theta.$$

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